Kantonsschule Im Lee

Schriftliche Maturitätsprüfungen 2018 im Fach Mathematik

Klasse:	4a
Profil:	M/MN/N
Lehrperson:	Rolf Kleiner
Zeit:	3 Stunden
Hilfsmittel:	Grafiktaschenrechner ohne CAS, Formelsammlung
Bemerkungen:	Die Prüfung enthält 8 Aufgaben mit 100 Punkten.
	Lösen Sie die Aufgaben in der zur Verfügung gestellten Broschüre.
	Schreiben Sie Ihre Lösungswege klar nachvollziehbar auf.
	Geben Sie numerische Ergebnisse, wenn möglich, exakt, andernfalls sinnvoll gerundet an.

1. [9m] Given is the function $f(x)=\frac{1}{6}x^3-2x^2+6x$.

a) Find both coordinates each of the maximum point, the minimum point and of the point of inflection of *f*.

b) Consider the function $g(x)=\frac{f(x)}{2(x-50)^3}=\frac{\frac{1}{6}x^3-2x^2+6x}{2(x-50)^3}$.

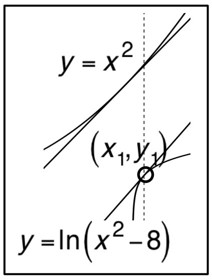
Give an equation each of the vertical asymptote and of the horizontal asymptote of the graph of *g*.

2. [13m] Given are the functions $f(x)=x^2$ and $g(x)=a\cdot x$ (with $a>0$).

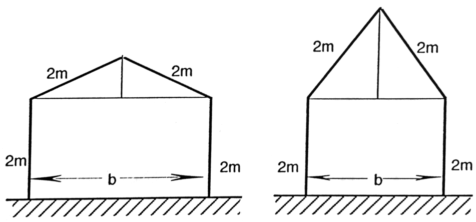
a) Find the value of *a*, if the area of the region enclosed by the graphs of *f* and *g* measures 36.

b) Let $a=6$. Therefore, $P(6,36)$ is a point of intersection of $f(x)=x^2$ and $g(x)=6\cdot x$. Find the angle between the two graphs at the point *P*.

c) At $x=x_1$, the tangents to $f(x)=x^2$ and to $y=\ln(x^2-8)$ are parallel (see figure, which shows part of the graphs). Find both coordinates of all such points (x_1,y_1) on the graph of $y=\ln(x^2-8)$.



3. [8m] A garden shed will be built with timber. A cross-section shows 4 beams, which are 2 meters long (see figure).



a) Show that the area of the cross-section in terms of the breadth *b* of the garden shed is $A=\frac{1}{4}b\left(8+\sqrt{16-b^2}\right)$.

b) Find the area of the largest possible cross-section.

4. [7m] The graph of $f(x)=0.3x-\cos(x)$ intersects the *x*-axis at x_0 with $0<x_0<\frac{\pi}{2}$.

a) Apply two steps of Newton's method to find an approximation of x_0.

b) The region enclosed by the graph of *f* for $0\le x\le x_0$ and the two coordinate-axes is rotated about the *x*-axis. Write an integral expression for the volume *V* of this solid of revolution and find an approximation of *V* by applying a numerical method of your choice (with any number of subintervals).

5. [16m] Given is the function $f(x)=\frac{x^2}{e^{2x}}$.

a) Show that $F(x)=\frac{-\left(2x^2+2x+1\right)\cdot e^{-2x}}{4}$ is an antiderivative of $f(x)$.

b) Find $\int_0^\infty f(x)dx$.

Given are $f'(x)=2x(1-x)e^{-2x}$ and $f''(x)=\left(4x^2-8x+2\right)e^{-2x}$.

c) Find the two local extremum points of the function $f(x)$. Carefully distinguish between the minimum and the maximum and give both coordinates of these points.

d) Explain why one of the points in part **c)** must be a *global* extremum point.

e) Find the interval(s) where the function $f(x)$ is concave down.

6. [15m] The point $A(1,10,10)$ is a vertex of the rectangle *ABCD*.

Let Π be the plane, in which the rectangle *ABCD* lies.

The straight line l: $\begin{pmatrix} x \\ y \\ z \end{pmatrix}=\begin{pmatrix} 6 \\ -1 \\ 14 \end{pmatrix}+t\cdot\begin{pmatrix} 1 \\ -4 \\ 8 \end{pmatrix}$ is normal to the plane Π and intersects the plane Π in vertex *B*.

a) Find a Cartesian equation of the plane Π.

b) Find the coordinates of the vertex *B*.

c) Show that the point $P(1,-10,0)$ lies in the plane Π.

d) Find the distance of the point $Q(6,0,10)$ from the plane Π.

e) Find the angle between the plane Π and the *x*-*y*-plane.

f) Find a vector equation (in parametric form) of the straight line, which passes through the vertices *B* and *C* of the rectangle *ABCD*.

g) Find the coordinates of the vertices *C* and *D* if the vertex *C* lies in the *x*-*y*-plane.

7. [16m] A sphere has the centre $M(4,0,-5)$ and passes through the point $P(6,-2,-6)$. Let the plane Π be the tangential plane to the sphere at *P*.

a) Find the radius of the sphere and give a Cartesian equation of the sphere.

b) Find a Cartesian equation of the tangential plane Π.

Further given is the straight line l: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -4 \\ 0 \\ 6 \end{pmatrix} + t \cdot \begin{pmatrix} 1 \\ 3 \\ -4 \end{pmatrix}$.

c) Show that the straight line l is parallel to the tangential plane Π.

d) The point L is the point on the straight line l which is closest to the sphere. Find the coordinates of the point L as well as the shortest distance to the sphere.

e) The sphere moves along a straight line parallel to the line l until it touches the sphere with the equation $(x-11)^2+(y-9)^2+(z+6)^2=36$. Find the coordinates of the centre of the moved sphere.

8. [16m] A group of students has developed a slot machine in the school's MINT laboratory. You pay 1 CHF and may then either win a first prize of 100 CHF or a consolation prize of 5 CHF or win nothing at all.
At first, a digit (0, 1, 2, … 9) is randomly chosen. If you are lucky enough to choose the correct digit, you will, in a second stage, win a consolation prize with a probability of 75%. However, if you do not choose the correct digit, you will, in the second stage, win a consolation prize with a probability of 5% only.
You may also, in the second stage, not win anything at all or then win a first prize with the probability of p. The value of p does not depend on the digit chosen in the first stage.

a) Draw a tree diagram to illustrate the situation and find the probability of winning a consolation prize.

If you have not been able to find the probability of winning a consolation prize you may use 0.12 for the probability of this event to solve the following problems.

b) Anna tries her luck 10 times. Find the probability that she
- **i)** never wins a consolation prize.
- **ii)** wins a consolation prize exactly twice.
- **iii)** wins a consolation prize at least twice.

c) Determine the number of times someone would have to play in order for the probability of winning a consolation prize at least once to become 99.5% or more.

d) The group of students who are running the slot machine would like to make a profit of 0.10 CHF on average per game. Determine the value of p (i.e. the probability of winning a first prize) in this case.

e) Let $p = 0.3\%$. Anna has heard that her friend has won a prize. Knowing this, determine the probability that it is a consolation prize only.

Vocabulary

1.	–	–
2.	garden shed	Gartenhaus
	timber	Holz / Holzbalken
	beam	Holzbalken / Balken
	cross-section	Querschnitt
3.	–	–
4.	–	–
5.	to distinguish	unterscheiden
6.	–	–
7.	–	–
8.	slot machine	Glücksspielautomat
	consolation prize	Trostpreis
	to try one's luck	sein Glück versuchen
	in order for	damit

Matura Exams 2018 in Mathematics – Class 4a N/MN/M immersive

Solutions, Grading Scale and Results

Bei komplizierteren Punkteverteilungen wird grundsätzlich pro groben Fehler ein Punkt abgezogen.

Falls exakte Resultate verlangt werden, gibt es auch für eine gerundete Dezimalzahl volle Punktzahl, falls ein (algebraischer) Lösungsweg ersichtlich ist.

1. **a)** $f'(x)=\frac{1}{2}x^2-4x+6$ 1↓ ; $f'(x)=0:\ x^2-8x+12=0$

$\Rightarrow x_1=2\ ;\ x_2=6$ 3↓ $\Rightarrow$ Max. $\left(2,\frac{16}{3}\right)$; Min. $(6,0)$ 5m

$\Rightarrow f''(x)=x-4$ 1↓ ; $f''(x)=0 \Rightarrow x=4 \Rightarrow$ point of inflection $\left(4,\frac{8}{3}\right)$ 2m

b) Vertical asymptote: denominator $=0$: $x=50$ 1m

Horizontal asymptote: $\lim\limits_{x\to\infty}\dfrac{\frac{1}{6}x^3-2x^2+6x}{2(x-50)^3}=\lim\limits_{x\to\infty}\dfrac{\frac{1}{6}x^3}{2x^3}=\dfrac{1}{12} \Rightarrow y=\dfrac{1}{12}$ 1m

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2. **a)** $x^2=ax$ 1↓ $\Rightarrow x=0$ or $x=a$ 2↓ $\Rightarrow A=\int_0^a\left(ax-x^2\right)dx$ 3↓

$=\left[\frac{a}{2}x^2-\frac{1}{3}x^3\right]_0^a$ 4↓ $=\frac{1}{6}a^3$ 5↓ $=36 \Rightarrow a=6$ 6m

b) $f'(x)=2x \Rightarrow f'(6)=12$ 1↓ ; $g'(x)=6$ +1m

$\Rightarrow \arctan(12)-\arctan(6)=$ 4.699° $=0.082$ rad 3m

c) $y'=\dfrac{2x}{x^2-8}$ 1↓ ; $\dfrac{2x}{x^2-8}=2x$ 2↓ $\Rightarrow x^2=9\ ;\ y=\ln(9-8)=0 \Rightarrow (\pm3,0)$ 4m

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3. **a)** $A=2b+\dfrac{b\cdot h}{2}$ 1↓ , $h^2+\left(\dfrac{b}{2}\right)^2=2^2 \Rightarrow h=\dfrac{\sqrt{16-b^2}}{2}$ +1m

$\Rightarrow A=2b+\dfrac{b\cdot\sqrt{16-b^2}}{4}=\dfrac{1}{4}b\left(8+\sqrt{16-b^2}\right)$ 3m

b) $A'(b)=\dfrac{1}{4}\left(8+\sqrt{16-b^2}+b\cdot\dfrac{-2b}{2\sqrt{16-b^2}}\right)$ 2↓ , $A'(b)=0:\ 8+\sqrt{}-\dfrac{b^2}{\sqrt{}}=0$

$\Rightarrow 8\sqrt{16-b^2}+\left(16-b^2\right)-b^2=0$ 3↓ $\Rightarrow 4\sqrt{16-b^2}=b^2-8$

$\Rightarrow 16\left(16-b^2\right)=b^4-16b^2+64$ 4↓ $\Rightarrow b=\sqrt[4]{192}\approx 3.722$

$\Rightarrow$ max. area of cross-section $A_{max}\approx 8.807$ 5m

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4. **a)** $x_{n+1}=x_n-\frac{0.3x_n-\cos(x_n)}{0.3+\sin(x_n)}$ 1↓ ; e.g. $x_1=1 \Rightarrow x_2=1.2105$ 2↓

$\Rightarrow$ $\boxed{x_3=1.2019\approx x_0}$ 3m

b) $\boxed{V=\pi\int_0^{1.2019}(0.3x-\cos(x))^2\,dx}$ 1↓ $\approx \pi\cdot\frac{b-a}{2\cdot 2}(y_0+2y_1+y_2)=$ +1m

$\pi\cdot\frac{1.2-0}{4}(1+2\cdot 0.415+0)$ 3↓ $=1.728 \Rightarrow$ more precise $\boxed{V\approx 1.672}$ 4m

or Simpson's Method with $n=2$: $\pi\cdot\frac{1.2-0}{6}(1+4\cdot 0.415+0)\approx 1.67$

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5. **a)** $F'(x)=-\frac{1}{4}\left((4x+2)e^{-2x}+(2x^2+2x+1)(-2)e^{-2x}\right)$ 3↓ $=-\frac{1}{4}(-4x^2)e^{-2x}=x^2e^{-2x}$ 4m

b) $\int_0^{\infty}f(x)dx=\left.\frac{-(2x^2+2x+1)\cdot e^{-2x}}{4}\right|_0^{\infty}$ 1↓ $=0-\left(\frac{-1\cdot 1}{4}\right)=$ $\boxed{\frac{1}{4}=0.25}$ 3m

c) $f'(x)=0:\ x=0$ or $x=1$ 1↓ ; $f''(0)>0$: min. at $x=0$; $f''(1)<0$: max. at $x=1$ 3↓

$\Rightarrow$ $\boxed{\text{min. point } (0,0)\text{; max. point } (1,e^{-2})}$ 4m

d) $\lim_{x\to-\infty}f(x)=\infty$, $\lim_{x\to\infty}f(x)=0$ and f continuous $\Rightarrow$ $\boxed{(0,0) \text{ is global min. point}}$ 2m

e) f is concave down $\Leftrightarrow f''(x)<0$; $f''(x)=0:\ 4x^2-8x+2=0$ 1↓ $\Rightarrow x=1\pm\frac{\sqrt{2}}{2}$ 2↓

$\Rightarrow$ $\boxed{f \text{ is concave down for } 1-\frac{\sqrt{2}}{2}<x<1+\frac{\sqrt{2}}{2} \text{ or } 0.293<x<1.707}$ 3m

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6. **a)** $\overrightarrow{n_\Pi}=\begin{pmatrix}1\\-4\\8\end{pmatrix}$ 1↓ ; $A\in\Pi:\Rightarrow$ $\boxed{\Pi:\ x-4y+8z-41=0}$ 2m

b) Point of intersection of plane Π and line l: $(6+t)-4(-1-4t)+8(14+8t)-41=0$ 1↓

$\Rightarrow t=-1 \Rightarrow$ $\boxed{B(5,3,6)}$ 2m

c) $P(1,-10,0)$ lies in the plane Π: $1+4\cdot 10+8\cdot 0-41=0$ 1m

d) $d(Q,\Pi)=\frac{|6-4\cdot 0+8\cdot 10-41|}{9}=\frac{45}{9}=5$ 2m

e) $\cos(\varphi)=\dfrac{\begin{pmatrix}1\\-4\\8\end{pmatrix}\circ\begin{pmatrix}0\\0\\1\end{pmatrix}}{9\cdot 1}$ 2↓ $\Rightarrow$ $\boxed{\varphi=27.266°}$ =0.4759 rad 3m

f) $\overrightarrow{AB}\times\overrightarrow{n_\Pi}$ 1↓ $=\begin{pmatrix}4\\-7\\-4\end{pmatrix}\times\begin{pmatrix}1\\-4\\8\end{pmatrix}=\begin{pmatrix}-72\\-36\\-9\end{pmatrix}$ 2↓ $\Rightarrow$ $\boxed{(BC):\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}5\\3\\6\end{pmatrix}+t\cdot\begin{pmatrix}8\\4\\1\end{pmatrix}}$ 3m

g) $z_C=0:\ 6+t=0\ \Rightarrow$ $\boxed{C(-43,-21,0)}$ 1m ;

$\overrightarrow{OD}=\overrightarrow{OC}+\overrightarrow{BA}\ =\begin{pmatrix}-43\\-21\\0\end{pmatrix}+\begin{pmatrix}-4\\7\\4\end{pmatrix}\ \Rightarrow$ $\boxed{D(-47,-14,4)}$ 1m

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7. a) $r=\overline{MP}=\sqrt{4+4+1}\ \Rightarrow$ $\boxed{r=3}$ 1↓ ; sphere: $\boxed{(x-4)^2+y^2+(z+5)^2=9}$ 2m

b) $\overrightarrow{n_\Pi}=\overrightarrow{MP}=\begin{pmatrix}2\\-2\\-1\end{pmatrix}$; $P\in\Pi:\Rightarrow$ $\boxed{\Pi:\ 2x-2y-z-22=0}$ 1m

c) $\overrightarrow{n_\Pi}\circ\vec{l}\overset{?}{=}0:\ \begin{pmatrix}1\\3\\-4\end{pmatrix}\circ\begin{pmatrix}2\\-2\\-1\end{pmatrix}=0$; $(-4,0,6)\overset{?}{\notin}\Pi:\ 2\cdot(-4)-2\cdot 0-6-22=-36\neq 0$

$\Rightarrow$ plane Π parallel to line l 2m

d) $L(-4+t,3t,6-4t)$ 1↓; $ML\perp l$: $\begin{pmatrix}-8+t\\3t\\11-4t\end{pmatrix}\circ\begin{pmatrix}1\\3\\-4\end{pmatrix}=0$ 2↓ $\Rightarrow\ t=2$ 3↓

$\Rightarrow$ $\boxed{L(-2,6,-2)}$ 4m ; shortest distance: $\overline{ML}-r=9-3\ \Rightarrow$ $\boxed{d=6}$ +1m

e) M' lies on a parallel line to l through M: $M'(4+t,3t,-5-4t)$. 1↓

$M_2(11,9,-6)$: $\overrightarrow{M'M_2}=\begin{pmatrix}7-t\\9-3t\\-1+4t\end{pmatrix}$ 2↓

M' lies at a distance of $r_1+r_2=3+6=9$ from M_2. 3↓

$\overline{M'M_2}=49-14t+t^2+81-54t+9t^2+1-8t+16t^2=9^2$ 4↓ $\Rightarrow\ 26t^2-76t+50=0$

$\Rightarrow\ t=\dfrac{76\pm 24}{52}\ \Rightarrow\ t=1$ (or $t=\dfrac{25}{13}>1$) 5↓ $\Rightarrow$ $\boxed{M'(5,3,-9)}$ 6m

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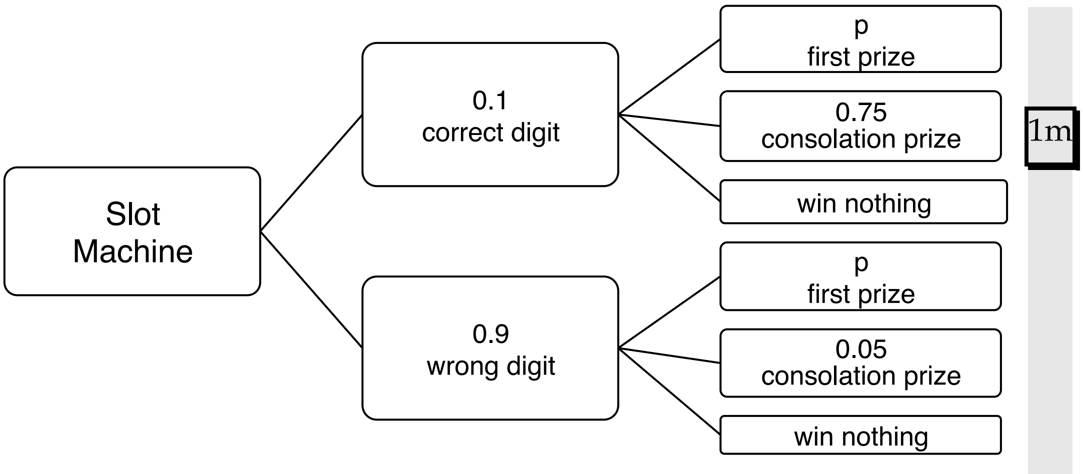
8. a) Tree diagram



$P(\text{consolation prize}) = 0.1 \cdot 0.75 + 0.9 \cdot 0.05 = \boxed{0.12}$ 1m

b) i) $0.88^{10} = \boxed{0.2785}$ 1m ; **ii)** $\binom{10}{2} \cdot (0.12)^2 \cdot (0.88)^8 = \boxed{0.2330}$ 2m ;

iii) $P(x \geq 2) = 1 - P(x = 0) - P(x = 1)$ 1↓ $= 1 - 0.2785 - 0.3798 = \boxed{0.3417}$ 2m

c) $1 - (0.88)^n > 0.995$ 1↓ $\Rightarrow n > \dfrac{\ln(0.005)}{\ln(0.88)} = 41.4$ 2↓ $\boxed{\text{play at least 42 times}}$ 3m

d)

X	0	5	100
P	...	0.12	p

$\Rightarrow E(X) = 0.6 + 100 \cdot p$ 2↓ $= 1 - 0.1$ CHF 3↓ or

X	1	$-5+1$	$-100+1$
P	$0.88 - p$	0.12	p

$\Rightarrow E(X) = 0.4 - 100 \cdot p$ 2↓ $= 0.1$ CHF 3↓

$\Rightarrow$ Probability of winning a first prize $\boxed{p = 0.003}$ 4m

e) $P(\text{consolation prize} \mid \text{any prize}) = \dfrac{0.12}{0.12 + 0.003} = \boxed{0.9756}$ 2m

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Grading Scale and Results – 4a N/MN/M immersive – Summer 2018											
Grade	**6**	**5.5**	**5**	**4.5**	**4**	**3.5**	**3**	**2.5**	**2**	**1.5**	**1.0**
number of marks	80	72	64	56	48	40	32	24	16	8	0
number of students	5	2	1	2	1	6	5	2	0	0	0

Problem	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**Total**
subject area	A	A	A	A	A	A	G	G	
number of marks	9	13	8	7	16	15	16	16	100
of which from pool	9	10	0	0	13	15	16	16	79