KANTONSSCHULE IM LEE
W I N T E R T H U R

MATURITÄTSPRÜFUNGEN 2015

Klasse: **4a**

Profile: M/MN/N

Lehrperson: Rolf Kleiner

MATHEMATIK
in englischer Sprache (immersiv)

Zeit: 3 Stunden
Time: 3 hours

Erlaubte Hilfsmittel: Grafiktaschenrechner ohne CAS, beliebige Formelsammlung
Approved resources: Graphing calculator without CAS, formula booklet of your choice

Bemerkungen: Die Prüfung enthält 8 Aufgaben mit 96 Punkten.
Remarks: The exam consists of 8 problems with 96 marks.

Lösen Sie jede Aufgabe auf ein separates A4-Blatt.
Solve each problem on a separate piece of paper.

Schreiben Sie Ihre Lösungswege klar nachvollziehbar auf.
Show all your working.

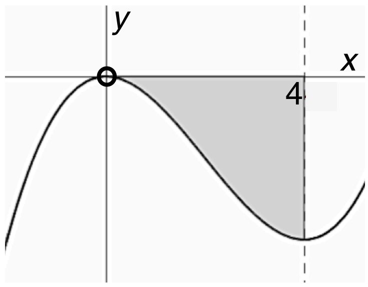
Geben Sie numerische Ergebnisse wenn möglich exakt, andernfalls sinnvoll gerundet an.
Give exact values for your numerical answers, if possible. Otherwise, round your results appropriately.

1. [6m] Given is the function $f(x) = \dfrac{x^2 - 3x}{x - 2}$.

a) Show that the function has neither maximum nor minimum points.

b) Find an equation of the tangent to the graph of f at $x = 3$.

2. [10m] The figure shows the graph of a polynomial function $y = f(x)$ of degree 3. The shaded area between the graph and the x-axis for $0 \le x \le 4$ measures $A = 64$. Read coordinates of critical points on the graph to find an equation of this function f.



3. [12m] Given is the function $f(x) = e^x - k$ (with $k > 1$). Let x_0 be its only root.

a) In order to find the root x_0, *one* step of Newton's Method with starting value $x_1 = \ln(k-1)$ is applied. Find an expression for x_2 and simplify it.

b) Find the exact value of the root x_0 in terms of k.

c) Use $\ln(10) \approx 2.303$ as well as the results above to find an approximation of $\ln(11)$ without using a calculator. Give your answer to 3 decimal places.

d) The area below the graph of $y = \dfrac{1}{x}$ for $10 \le x \le 11$ can be approximated by the area of a circumscribed rectangle. Draw a rough figure and show that this leads to the same approximation as in part **c)**.

4. [13m] Consider the region R enclosed by the graph of the function $y = \cos(3x)$ and the x-axis on the interval $0 \le x \le \dfrac{\pi}{6}$.

a) Find the area of the region R.

The vertical straight line $x = a$ splits the region R into two regions R_1 ($0 \le x \le a$) and R_2 ($a \le x \le \dfrac{\pi}{6}$) of equal area.

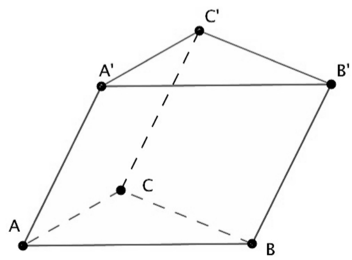
b) Find the exact value of a.

The regions R_1 and R_2 are now rotated about the x-axis. The solids of revolution formed in this way have volumes of V_1 and V_2, respectively.

c) Explain without performing any further calculations whether V_1 is less than V_2, equal to V_2 or greater than V_2.

d) Write an integral expression for V_1 and then, find an approximation of V_1 by using Simpson's Rule (parabola method) with 6 subintervals. You may assume $a \approx 0.18$.

5. [16m] A three-sided oblique prism has as bases the triangles ABC and $A'B'C'$. Given are the vertices $A(4,2,1)$, $B(5,0,-1)$, $C(1,4,5)$ and $A'(1,-8,8)$, see figure (not to scale).



a) Find the coordinates of the vertex B'.

b) Find a Cartesian equation of the plane (ABC).

c) Find the angle between the edge AA' and the base (ABC).

d) Find the volume of the prism.

e) A three-sided *pyramid* with the same base ABC has the volume $V = 18$. Its apex lies on the straight line (PQ) with $P(0,0,4)$ and $Q(5,-5,10)$. Find the coordinates of all possible apexes.

6. [17m] Of a sphere, the centre $O(4,1,2)$ as well as the point $A(10,9,2)$ on the surface of the sphere are given. The straight line $l = (AB)$ with $B(13,13,-3)$ intersects the sphere in the given point $A(10,9,2)$ and in a second point S.

a) Find a Cartesian equation of the sphere.

b) Find the coordinates of the second point of intersection S of the line $l = (AB)$ and the sphere.

Let Π be the tangent plane to the sphere in the point A.

c) Find an equation of the tangent plane Π.

d) Given is the point $C(4,1,-8)$.

i) Show that the point C lies on the surface of the sphere.

ii) Show that the angle $\varphi = \sphericalangle(AOC)$ equals 90°.

iii) Find the distance between the point C and the tangent plane Π.

e) Given is a further plane $\Pi_2 : 3x + 4y - 5z - 56 = 0$.

i) Show that the point $A(10,9,2)$ (as above) also lies in plane Π_2.

ii) Find an equation of the line of intersection of the planes Π and Π_2.

f) Given is the point $P(7,5,7)$.

i) Show that the point P lies on the *inside* of the sphere.

ii) Find a Cartesian equation of the plane Π_3, which contains P and intersects the sphere in the smallest possible circle.

7. [22m] A bag contains 9 white balls and 2 black balls.

a) Amy randomly draws a ball, puts it back into the bag and repeats this 20 times. Find the probability that Amy draws 15 white and 5 black balls.

b) Betty draws *one* ball. If it is white she wins 10 pounds. If it is black she loses t pounds. Let X denote the number of pounds Betty wins.

i) Find (without necessarily simplifying your results) the expected value and the standard deviation of X in terms of t.

ii) Determine the value of t so that the game is fair.

A second bag contains 3 white balls and 7 black balls.
Cate and Dana play a game where the first person to draw two white balls wins. On one's turn, firstly, one of the two bags is chosen at random, then two balls are simultaneously drawn from the chosen bag and, finally, put back into the bag. Cate starts.

c) Show that the probability that Cate wins the game on her first turn is $p \approx 0.36$.

Use the result of part **c)** to solve parts **d)**, **e)** and **f)**.

d) Cate wins the game on her first turn. Find the probability that, in this case, the bag she chose was the *first* bag.

e) Find the probability that Dana wins the game on her first turn.

f) Find the probability that Cate wins the game. Hint: Consider a geometric series.

Emma is allowed to remove a few white balls from the first bag and place them into the second bag in order to increase her chances of drawing a white ball. Let n be the total number of white balls ending up in the first bag. After this, Emma chooses one of the bags at random and then draws *one* ball from it.

g) Show that the probability of drawing a white ball can be expressed as $P(n) = \frac{1}{2} \cdot \frac{2n^2 - 29n - 24}{n^2 - 17n - 38}$ and hence, using your graphing calculator, determine how many white balls should be in each bag so that Emma's chances of drawing a white ball are as high as possible. Carefully show how you use your graphing calculator.

Vocabulary

1.	neither … nor	weder ... noch
2.	polynomial function of degree 3	ganz-rationale Funktion 3. Grades
3.	root (or zero or *x*-intercept) of *f*	Nullstelle von *f*
	circumscribed rectangle (*as opposed to* ‚inscribed')	umschriebenes Rechteck (*im Gegensatz zu* ‚einbeschrieben')
4.	–	
5.	three-sided oblique prism	dreiseitiges, schiefes Prisma
	bases	Grund- und Deckfläche
	not to scale	nicht massstabsgetreu
	apex (*plural:* apexes) of a pyramid	Spitze (*Plural:* Spitzen) einer Pyramide
6.	tangent plane	Tangentialebene
7.	On one's turn, …	Wenn eine der Spielerinnen an der Reihe ist, …

1. a) $f'(x)=\dfrac{(2x-3)(x-2)-1\cdot(x^2-3x)}{(x-2)^2}$ $\boxed{1\downarrow}$ $=\dfrac{x^2-4x+6}{(x-2)^2}\Rightarrow f'(x)=0:\ x^2-4x+6=0$ $\boxed{2\downarrow}$

$b^2-4ac=16-24<0\Rightarrow f'(x)=0$ has no solution $\Rightarrow f$ has no extremum $\boxed{3\text{m}}$

b) $f'(3)=3$ $\boxed{1\downarrow}$; $f(3)=0\Rightarrow$ $\boxed{y=3x-9}$ $\boxed{3\text{m}}$

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2. $f(x)=ax^3+bx^2+cx+d$; $f(0)=0\Rightarrow d=0$ $\boxed{1\text{m}}$; $f'(x)=3ax^2+2bx+c$ $\boxed{1\text{m}}$;

$f'(0)=0\Rightarrow c=0$ $\boxed{+1\text{m}}$; $f'(4)=0\Rightarrow 48a+4b=0$ $\boxed{+1\text{m}}$ $\Rightarrow b=-6a$;

$\int_0^4 f(x)\,dx=-64$ $\boxed{+2\text{m}}$ $\Rightarrow \int_0^4\left(ax^3-6ax^2\right)dx=\left[\dfrac{a}{4}x^4-2ax^3\right]_0^4$ $\boxed{+1\text{m}}$ $\Rightarrow 64a-128a=-64$

$\Rightarrow a=1$ $\boxed{8\downarrow}$ $\Rightarrow b=-6$ $\boxed{9\downarrow}$ $\Rightarrow$ $\boxed{f(x)=x^3-6x^2}$ $\boxed{10\text{m}}$

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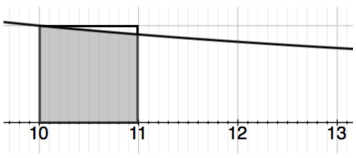
3. a) $f'(x)=e^x\Rightarrow x_{n+1}=x_n-\dfrac{e^{x_n}-k}{e^{x_n}}$ $\boxed{2\downarrow}$; $x_1=\ln(k-1)$:

$x_2=\ln(k-1)-\dfrac{(k-1)-k}{k-1}$ $\boxed{3\downarrow}$ $\Rightarrow$ $\boxed{x_2=\ln(k-1)+\dfrac{1}{k-1}}$ $\boxed{4\text{m}}$

b) $e^{x_0}-k=0\Rightarrow$ $\boxed{x_0=\ln(k)}$ $\boxed{1\text{m}}$

c) $x_0\approx x_2$: $\ln(11)\approx\ln(10)+\dfrac{1}{10}$ $\boxed{1\downarrow}$ $=2.303+0.1\Rightarrow$ $\boxed{2.403}$ $\boxed{2\text{m}}$ ($\ln(11)=2.398$)

d) 

$\boxed{1\text{m}}$ $A_{\text{curve}}=\int_{10}^{11}\dfrac{1}{x}dx=\ln(11)-\ln(10)$ $\boxed{2\text{m}}$; $A_{\text{rectangle}}=\dfrac{1}{10}$ $\boxed{1\text{m}}$

$\Rightarrow \ln(11)-\ln(10)\approx\dfrac{1}{10}\Rightarrow\ln(11)\approx\ln(10)+\dfrac{1}{10}$ $\boxed{5\text{m}}$

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4. a) $\int_0^{\pi/6}\cos(3x)\,dx$ $\boxed{1\downarrow}$ $=\dfrac{1}{3}\left[\sin(3x)\right]_0^{\pi/6}$ $\boxed{2\downarrow}$ $=$ $\boxed{\dfrac{1}{3}}$ $\boxed{3\text{m}}$

b) $\int_0^a\cos(3x)\,dx=\dfrac{1}{6}$ $\boxed{1\downarrow}$ $\Rightarrow\dfrac{1}{3}\cdot\sin(3a)=\dfrac{1}{6}$ $\boxed{2\downarrow}$ $\Rightarrow 3a=\arcsin\left(\dfrac{1}{2}\right)\Rightarrow$ $\boxed{a=\dfrac{\pi}{18}\approx 0.175}$ $\boxed{3\text{m}}$

c) $V_1>V_2$ because, on average, R_1 is farther away from the x-axis than R_2. $\boxed{2\text{m}}$

d) $V_1 = \pi \cdot \int_0^{0.18} (\cos(3x))^2\, dx$ 1↓

$\approx \pi \cdot \frac{0.18-0}{3\cdot 6} \cdot [\cos^2(0)+4\cos^2(0.09)+2\cos^2(0.18)+4\cos^2(0.27)+2\cos^2(0.36)+$

$+4\cos^2(0.45)+\cos^2(0.54)]$ 3↓ $\Rightarrow \boxed{V_1 \approx 0.1635\pi \approx 0.5136}$ 5m (exact: 0.50, V_2=0.32)

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5. a) $\overrightarrow{OB'} = \overrightarrow{OA'} + \overrightarrow{AB} = \begin{pmatrix} 1 \\ -8 \\ 8 \end{pmatrix} + \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} = \overrightarrow{OB} + \overrightarrow{AA'} = \begin{pmatrix} 5 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} -3 \\ -10 \\ 7 \end{pmatrix} \Rightarrow \boxed{B'(2,-10,6)}$ 1m

b) $\overrightarrow{AB} \times \overrightarrow{AC} = \begin{pmatrix} 1 \\ -2 \\ -2 \end{pmatrix} \times \begin{pmatrix} -3 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} -4 \\ 2 \\ -4 \end{pmatrix} \Rightarrow \vec{n} = \begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ 2↓ Plane: $\boxed{2x-y+2z-8=0}$ 3m

c) $sin(\varphi) = \frac{\vec{n} \circ \overrightarrow{AA'}}{|\vec{n}| \cdot |\overrightarrow{AA'}|}$ 1↓ $= \frac{18}{3 \cdot \sqrt{158}} \approx 0.477 \Rightarrow \boxed{\varphi \approx 28.51^\circ}$ 3m

d) height = $d(A',(ABC)) = \frac{2\cdot 1-(-8)+2\cdot 8-8}{\sqrt{2^2+(-1)^2+2^2}} = 6$ +2m

Base G = $0.5 \cdot |\overrightarrow{AB} \times \overrightarrow{AC}| = 0.5 \cdot 6 = 3$ +2m ; $V_{prism} = G \cdot h \Rightarrow \boxed{V = 18}$ 4m

e) $V = \frac{1}{3} \cdot G \cdot h \Rightarrow 18 = \frac{1}{3} \cdot 3 \cdot h \Rightarrow h = 18$ +1m ; point on line $(5t, -5t, 4+6t)$ +1m

$\frac{2\cdot 5t-(-5t)+2\cdot(4+6t)-8}{3} = \pm 18$ 3↓ $\Rightarrow t = \pm 2 \Rightarrow \boxed{(-10,10,-8), (10,-10,16)}$ 5m

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6. a) $r = \overline{AM} = \sqrt{36+64+0} = 10$ 1↓ $\Rightarrow \boxed{(x-4)^2+(y-1)^2+(z-2)^2 = 100}$ 2m

b) point on line $(10+3t, 9+4t, 2-5t) \Rightarrow (6+3t)+(8+4t)^2+(-5t)^2 = 100$ 2↓

$\Rightarrow t(t+2) = 0$; $t = -2$: $\boxed{S(4, 1, 12)}$ 3m

c) $\vec{n} = \overrightarrow{OA} = \begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix} \Rightarrow$ tangent plane: $\boxed{3x+4y-66=0}$ 2m

d) $(4-4)^2+(1-1)^2+(-8-2)^2 = 100$ ✓ 1m ; $\overrightarrow{OA} \times \overrightarrow{OC} = \begin{pmatrix} 6 \\ 8 \\ 0 \end{pmatrix} \circ \begin{pmatrix} 0 \\ 0 \\ -10 \end{pmatrix} = 0 \Rightarrow \varphi = 90^\circ$ 1m

$\frac{|3\cdot 4+4\cdot 1-66|}{\sqrt{3^2+4^2}} = 10 \Rightarrow d(C,\Pi) = 10 = r$ 1m

e) $3\cdot10+4\cdot9-5\cdot2-56=0$ ✓ $\Rightarrow A\in\Pi_2$ $\boxed{1m}$; direction vector of $l=\overrightarrow{n_{\Pi}}\times\overrightarrow{n_{\Pi_2}}$;

$$\Rightarrow \begin{pmatrix}3\\4\\0\end{pmatrix}\times\begin{pmatrix}3\\4\\-5\end{pmatrix}=\begin{pmatrix}-20\\15\\0\end{pmatrix}\Rightarrow \boxed{l:\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}10\\9\\2\end{pmatrix}+t\cdot\begin{pmatrix}4\\-3\\0\end{pmatrix}}\ \boxed{3m}$$

f) $(7-4)^2+(5-1)^2+(7-2)^2=50<100=r^2 \Rightarrow P$ lies inside of the sphere $\boxed{1m}$;

$$\overrightarrow{n_{\Pi_3}}=\overrightarrow{OP}=\begin{pmatrix}3\\4\\5\end{pmatrix}\Rightarrow \text{Plane } \Pi_3: \boxed{3x+4y+5z-76=0}\ \boxed{2m}$$

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7. a) $B_{20,\frac{9}{11}}(X=15)\ \boxed{1\downarrow} = \boxed{0.1518\approx15.18\%}\ \boxed{2m}$

b) i) $E(X)=10\cdot\frac{9}{11}+t\cdot\frac{2}{11}=\boxed{\frac{2t+90}{11}}\ \boxed{2m}$; $\boxed{\sigma(X)=\sqrt{10^2\cdot\frac{9}{11}+t^2\cdot\frac{2}{11}-\left(\frac{2t+90}{11}\right)^2}}\ \boxed{2m}$

ii) The game is fair if $E(X)=0 \Rightarrow$ $\boxed{t=(-)45 \text{ pounds}}\ \boxed{1m}$

c) $p=\frac{1}{2}\cdot\frac{9}{11}\cdot\frac{8}{10}+\frac{1}{2}\cdot\frac{3}{10}\cdot\frac{2}{9}\ \boxed{2\downarrow} = \boxed{\frac{119}{330}=0.3606}\ \boxed{3m}$

d) $P(\text{first bag if Cate wins})=\frac{P(\text{first bag and Cate wins})}{P(\text{Cate wins})}=\frac{\frac{1}{2}\cdot\frac{9}{11}\cdot\frac{8}{10}}{\frac{119}{330}}\ \boxed{0.90756}\ \boxed{2m}$

e) $P(\text{Dana wins on first turn})=(1-p)\cdot p\ \boxed{1\downarrow} = \boxed{0.23057}\ \boxed{2m}$

f) $P(\text{Cate wins})=p+(1-p)^2\cdot p+(1-p)^4\cdot p+\ldots\ \boxed{2\downarrow} \Rightarrow s_\infty=\frac{a_1}{1-q}=\frac{p}{1-(1-p)^2}=\ \boxed{3\downarrow}$

$\Rightarrow \boxed{P(\text{Cate wins})=0.60998\approx0.61}\ \boxed{4m}$

g) $P(n)=\frac{1}{2}\cdot\frac{n}{n+2}+\frac{1}{2}\cdot\frac{12-n}{19-n}\ \boxed{1\downarrow} = \frac{1}{2}\cdot\left(\frac{n(19-n)+(12-n)(n+2)}{(n+2)(19-n)}\right)=\boxed{\frac{1}{2}\cdot\frac{2n^2-29n-24}{n^2-17n-38}}\ \boxed{2m}$

e.g. calculate max. of $P(n)$: $(5.315, 0.6071) \Rightarrow$ $\boxed{\text{5 whites in bag 1; 7 in bag 2}}\ \boxed{2m}$

Grading Scale and Results – 4a (Immersion) – Summer 2015											
Grade	**6**	**5.5**	**5**	**4.5**	**4**	**3.5**	**3**	**2.5**	**2**	**1.5**	**1.0**
Number of Marks	80	72	64	56	48	40	32	24	16	8	0
Number of Students	2	2	1	5	2	3	2	2	1	0	0