KANTONSSCHULE IM LEE
WINTERTHUR

MATURITÄTSPRÜFUNGEN 2014

Klasse:	**4g**
Profile:	A/M/MN/N
Lehrperson:	Rolf Kleiner

MATHEMATIK
in englischer Sprache (immersiv)

Zeit:	3 Stunden *3 hours*
Erlaubte Hilfsmittel:	Grafiktaschenrechner ohne CAS, beliebige Formelsammlung *Graphing calculator without CAS, formula booklet of your choice*
Bemerkungen:	Die Prüfung enthält 8 Aufgaben mit 96 Punkten. *The exam consists of 8 problems with 96 marks.* Lösen Sie jede Aufgabe auf ein separates A4-Blatt. *Solve each problem on a separate piece of paper.* Schreiben Sie Ihre Lösungswege klar nachvollziehbar auf. *Show all your working.* Geben Sie numerische Ergebnisse wenn möglich exakt, andernfalls sinnvoll gerundet an. *Give exact values for your numerical answers, if possible. Otherwise, round your results appropriately.*

1. [9m] Given is the function $f(x) = x \cdot \ln(x) - x^2$.

a) Find an expression for $f'(x)$, $f''(x)$, $f'''(x)$, $f^{(4)}(x)$ and $f^{(5)}(x)$ and deduce an expression for the *n*-th derivative of the function *f* for $n \geq 3$.

b) Find the *x*-coordinate of the point on the curve of *f* for which the curve is the least steep.

2. [15m] Given is the function $f(x) = e^{(x^2)} + e^{-(x^2)} - 4$. It has *x*-intercepts at $x \approx \pm 1.15$.

a) Show that two steps of Newton's Method with starting value $x_1 = 1$ lead to a good approximation of the positive *x*-intercept of *f*.

Let *A* be the area of the region enclosed between the curve of *f* and the *x*-axis.

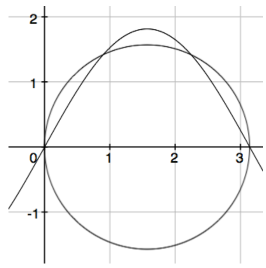
b) Find an approximation of *A* by applying Simpson's Rule (parabola method) with strips of width $\Delta x = \frac{1.15}{2} = 0.575$.

c) Find the value of *c* such that the curve of $y = c \cdot x^4 - 2$ also has *x*-intercepts at $x = \pm 1.15$. Then, use this result to find an approximation for the area *A*.

3. [9m] Consider the curve of $y = k \cdot \sin(x)$ for $0 \leq x \leq \pi$.

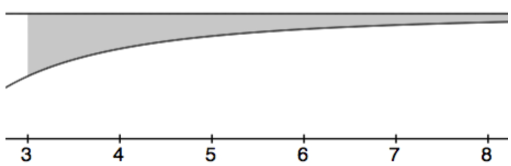
a) Find the *x*-value for which the value of the gradient of this curve is <u>half</u> of the value of the gradient at $x = \frac{\pi}{8}$.

b) The region enclosed between the curve of $y = k \cdot \sin(x)$ and the *x*-axis is rotated about the *x*-axis to form a solid of revolution. The volume of this solid is to be equal to the volume of a sphere with radius $\frac{\pi}{2}$ (see figure).

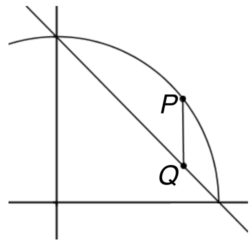


Find the value of the number *k* by using your graphing calculator to evaluate the necessary integral.

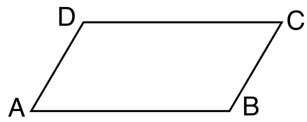
4. [6m] Find the area of the unbounded region between the curve of $y = \frac{2x^2 - 9}{x^2}$ and its horizontal asymptote for $x \geq 3$ by using an antiderivative (see figure, not to scale).



5. [8m] The point P lies on the semicircle $y = \sqrt{1-x^2}$ and the point Q has the same x-coordinate as P but lies on the straight line through the points (0,1) and (1,0) (see figure). Find a simplified expression for the exact value of the greatest possible length of the vertical line segment PQ by using differential calculus.



6. [16m] The points $A(1,1,1)$, $B(1,4,5)$ and $C(1,-1,-1)$ are vertices of the parallelogram $ABCD$ (see figure).



a) Find the coordinates of the point D.

b) Find the coordinates of the point of intersection E of the two diagonals.

c) Let F be the point on the straight line (AB) which is closest to the point C. Determine whether F lies <u>between</u> the two points A and B or whether F lies <u>outside</u> of the line segment AB.

d) Let $ABCDS$ be a pyramid with perpendicular height SE. Reminder: E is the point of intersection of the diagonals of the parallelogram - see part **b)**. Find possible coordinates of the point S if the volume of the pyramid is 6.

7. [16m] Let S be a sphere with centre $O(1,0,5)$ and let $P(4,0,1)$ be a point on the surface of the sphere.

a) Give a Cartesian equation of the sphere.

b) The straight line (AB) is given through the points $A(4,2,3)$ and $B(4,4,5)$.

Find the points of intersection of this straight line with the sphere.

c) Find a Cartesian equation of the tangential plane to the sphere in the point P.

d) Find the angle between the straight line (OP) and the y-z-plane.

e) The sphere S intersects the y-z-plane in a circle. Find the centre and the radius of this circle.

8. [17m] Otto is a student who, unfortunately, cheats on tests. The probability that he gets caught is $p_F = 1{:}10$ in each of the 7 French tests and $p_M = ?$ in each of the 9 Maths tests of the school year.

a) Find the probability that Otto will be caught in
i) none of the 7 French tests, **ii)** in at least one of the 7 French tests.

b) Determine p_M so that the probability of not getting caught in any of the 9 Maths tests is greater than 80%.

c) In French, 3 hours of extra work are assigned for each test on which a student is found cheating. In Maths, it is 2 hours of extra work. Find the expected total number of hours of extra work for Otto in French and Maths in the course of the school year if $p_M = 0.15$.

Let p be the probability that a randomly chosen student has been cheating.

d) Assume $p = 0.25$. Find the probability that in a randomly chosen class with 18 students not more than two students have been cheating.

At *Im Luv High School* an anonymous survey was conducted to find an estimate of p. Students were asked to roll a die secretly and then, if the die showed 5 or 6, to give an untrue answer, i.e. to *say* that they had been cheating although they had always been honest, and vice versa. If the die showed 1, 2, 3 or 4, however, the students were asked to answer the question truthfully. It is assumed that all students followed these instructions.

e) Explain, in a sentence, the purpose of throwing a die before answering.

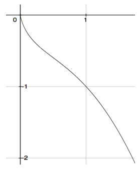
f) 75 out of 180 students replied that they had been cheating. Show, using a tree diagram, that this leads to an estimate of $p = 0.25$.

g) Find the probability, using the information of part **f)**, that a student who *said* that he had been cheating *actually* never cheated.

End of Examination

1. a) $\boxed{f'(x)=1\cdot\ln(x)+x\cdot\frac{1}{x}-2x=\ln(x)+1-2x}$ 2↓ $\Rightarrow \boxed{f''(x)=\frac{1}{x}-2}$ 3↓ $\Rightarrow \boxed{f'''(x)=-x^{-2}}$ 4↓

$\Rightarrow \boxed{f^{(4)}(x)=2x^{-3}\text{ , } f^{(5)}(x)=-6x^{-4}}$ 5↓ $\Rightarrow \boxed{f^{(n)}(x)=\frac{(-1)^n\cdot(n-2)!}{x^{n-1}}}$ 7m



b) $y''=0:\ \frac{1}{x}-2=0 \Rightarrow \boxed{x=0.5}$ 2m

---

2. a) $f'(x)=2xe^{(x^2)}-2xe^{-(x^2)}$ 2↓ ; $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)} \Rightarrow x_{n+1}=x_n-\frac{e^{(x_n^2)}+e^{-(x_n^2)}-4}{2x_n\cdot e^{(x_n^2)}-2x_n\cdot e^{-(x_n^2)}}$ 4↓

$x_1=1 \Rightarrow x_2=1.1944$ 5↓ $\Rightarrow x_3=1.1512\ \ (\ldots\Rightarrow x=1.1476) \Rightarrow \boxed{x_n\approx 1.15}$ 6m

b) $A\approx 2\cdot\left|\frac{1.15-0}{3\cdot 2}\cdot\left(-2+4\left(e^{0.575^2}+e^{-0.575^2}-4\right)+0\right)\right|$ 3↓ $=2\cdot|-1.832| \Rightarrow \boxed{A\approx 3.66}$ 4m

c) $c\cdot 1.15^4-2=0$ 1↓ $\Rightarrow c=1.14$ 2↓ ; $A\approx 2\cdot\left|\int_0^{1.15}(1.14x^4-2)dx\right|$ 3↓ $\Rightarrow \boxed{A\approx 3.68}$ 5m

---

3. a) $f'\left(\frac{\pi}{8}\right)=2\cdot f'(x)$ 1↓ $\Rightarrow k\cdot\cos\left(\frac{\pi}{8}\right)=2k\cdot\cos(x)$ 2↓ $x=\arccos\left(\frac{0.924}{2}\right) \Rightarrow \boxed{x=1.09}$ 4m

b) $V_{revol.}=\pi\cdot\int_0^{\pi}(k\cdot\sin(x))^2dx$ 1m ; $V_{sphere}=\frac{4\pi}{3}\cdot\left(\frac{\pi}{2}\right)^3=16.235$ or $\frac{1}{6}\pi^4$ 1m ;

$\Rightarrow \pi\cdot k^2\cdot\int_0^{\pi}\sin^2(x)\,dx=16.235$ 3↓ $\Rightarrow \pi\cdot k^2\cdot 1.57=16.235 \Rightarrow \boxed{k=\pm\tfrac{\sqrt{3}}{3}\pi\approx\pm 1.8}$ 5m

---

4. Horizontal asymptote: $y=2$ 1↓ ; Area $=\int_3^{\infty}\left(2-\frac{2x^2-9}{x^2}\right)dx$ 3↓ $=\int_3^{\infty}\frac{9}{x^2}dx$ 4↓

$=\left[-\frac{9}{x}\right]_3^{\infty}$ 5↓ $=0-(-3) \Rightarrow \boxed{A=3}$ 6m

---

5. Straight line $y=1-x$ 1↓ ; Goal function $\overline{PQ}=f(x)=\sqrt{1-x^2}-(1-x)$ 2↓

$\Rightarrow f'(x)=\frac{-2x}{2\sqrt{1-x^2}}+1$ 4↓ ; $f'(x)=0:\ \frac{-x}{\sqrt{1-x^2}}+1=0$ 5↓ $\Rightarrow \sqrt{1-x^2}=x \Rightarrow x=\frac{1}{\sqrt{2}}$ 6↓

$\Rightarrow \overline{PQ}=\sqrt{1-\frac{1}{2}}-\left(1-\frac{1}{\sqrt{2}}\right) \Rightarrow \boxed{\overline{PQ}=\sqrt{2}-1}$ 8m

---

6. **a)** $\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{BC}$ 1↓ $=\begin{pmatrix}1\\1\\1\end{pmatrix}+\begin{pmatrix}0\\-5\\-6\end{pmatrix}\Rightarrow \boxed{D(1,-4,-5)}$ 2m

b) $E=M_{AC}=\left(\dfrac{x_A+x_C}{2},\dots\right)$ 1↓ $\Rightarrow \boxed{E(1,0,0)}$ 2m

c) Normal plane to AB through C: $3y+4z+7=0$ 2m ; line (AB): $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}1\\1\\1\end{pmatrix}+t\begin{pmatrix}0\\3\\4\end{pmatrix}$ 1m

Intersection F of line with plane: $3(1+3t)+4(1+4t)+7=0\ \Rightarrow t=-\dfrac{14}{25}<0$ 5↓

$\Rightarrow$ because $t\notin[0,1]$: $\boxed{\text{point } F \text{ lies outside of the line segment } AB}$ 6m

d) $V=\dfrac{G\cdot h}{3}=6\Rightarrow h=\overline{SE}=\dfrac{18}{G}$ 1m ; $G=\left|\overrightarrow{AB}\times\overrightarrow{AD}\right|$ 1↓ $=\left|\begin{pmatrix}0\\3\\4\end{pmatrix}\times\begin{pmatrix}0\\-5\\-6\end{pmatrix}\right|$ 2↓ $=\left|\begin{pmatrix}2\\0\\0\end{pmatrix}\right|=2$ 3↓

$\Rightarrow h=\dfrac{18}{G}=9$ 4↓ ; with $E(1,0,0)$: $\overrightarrow{OS}=\begin{pmatrix}1\\0\\0\end{pmatrix}\pm 9\cdot\begin{pmatrix}1\\0\\0\end{pmatrix}\Rightarrow \boxed{S_1(10,0,0)\ ,\ S_2(-8,0,0)}$ 6m

---

7. **a)** $r=\overline{OP}=\sqrt{3^2+0^2+(-4)^2}=5\Rightarrow \boxed{(x-1)^2+y^2+(z-5)^2=25}$ 2m

b) line (AB): $\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}4\\2\\3\end{pmatrix}+t\begin{pmatrix}0\\2\\2\end{pmatrix}$ 1↓ $\Rightarrow$ intersect with S: $3^2+(2+2t)^2+(-2+2t)^2=25$ 2↓

$\Rightarrow 8t^2=8\ \Rightarrow t=\pm 1$ 3↓ $\Rightarrow$ points of intersection: $\boxed{D_1(4,4,5)\ ,\ D_2(4,0,1)}$ 4m

c) normal vector $\overrightarrow{OP}=\begin{pmatrix}3\\0\\-4\end{pmatrix}\Rightarrow$ tangential plane: $\boxed{3x-4z-8=0}$ 2m

d) $\varphi=\arcsin\dfrac{\left|\overrightarrow{OP}\circ\overrightarrow{n_{yz}}\right|}{\left|\overrightarrow{OP}\right|\cdot\left|\overrightarrow{n_{yz}}\right|}$ 1↓ $=\arcsin\dfrac{\left|\begin{pmatrix}3\\0\\-4\end{pmatrix}\circ\begin{pmatrix}1\\0\\0\end{pmatrix}\right|}{5\cdot 1}=\arcsin\dfrac{3}{5}$ 3↓ $=\boxed{36.8699^\circ}$ 4m

e) y-z-plane: $x=0$ 1↓ ; intersection with S: $(0-1)^2+y^2+(z-5)^2=25$ 2↓

$\Rightarrow y^2+(z-5)^2=24\Rightarrow \boxed{\text{centre } (0,0,5)\text{ , radius } r=\sqrt{24}\approx 4.8990}$ 4m

---

8. a) **i)** $0.9^7 = \boxed{0.4783 \approx 47.8\%}$ 1m ; **ii)** $1-0.9^7 = \boxed{0.5217 \approx 52.2\%}$ 1m

b) $(1-p)^9 > 0.80 \;\Rightarrow\; p < 1-\sqrt[9]{0.80} = 0.02449$

$\Rightarrow$ $\boxed{\text{If } p < 2.4\%\text{, then } P(\text{never getting caught}) > 80\%}$ 3m

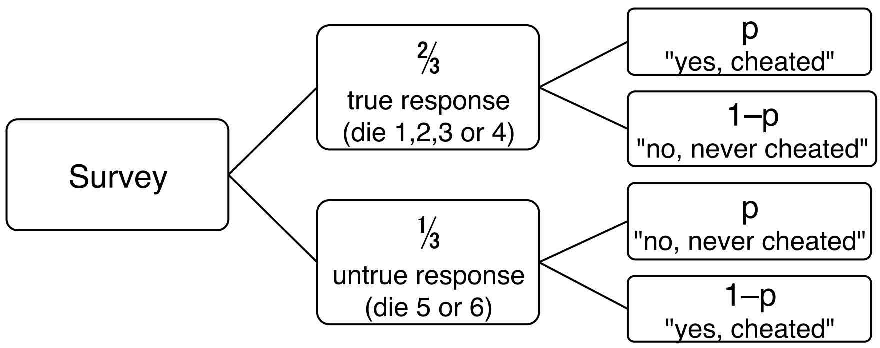
c) $E = 7\cdot 3\cdot \frac{1}{10} + 9\cdot 2\cdot 0.15 = 2.1 + 2.7 \Rightarrow$ $\boxed{4.8\text{ h extra work to be expected}}$ 2m

d) $B_{18,0.25}(X \le 2)$ 1↓ $= 0.75^{18} + 18\cdot 0.25\cdot 0.75^{17} + \binom{18}{2}\cdot 0.25^2\cdot 0.75^{16}$ 2↓

$= 0.005638 + 0.033826 + 0.095841 = \boxed{0.1353 = 13.53\%}$ 3m

e) Students know that their answers do not allow any conclusions to be drawn regarding their true actions. So, even if a student might not want to admit to cheating, he may – in this survey – hesitate less to say: „Yes, I have been cheating“, because he knows that this answer does not mean that he actually has been cheating. 1m

f) 2↓



$\frac{2}{3}p + \frac{1}{3}(1-p) = \frac{75}{180}$ 3↓ $\Rightarrow \frac{1}{3}p = \frac{75}{180} - \frac{1}{3} = \frac{1}{12} \Rightarrow \boxed{p = 0.25}$ 4m

g) $\dfrac{\frac{1}{3}\cdot\frac{3}{4}}{\frac{75}{180}} = 0.6 \Rightarrow$ $\boxed{\text{60\% who \textit{said} “yes” actually never cheated.}}$ 2m

---

Grading Scale and Results – 4g (Immersion) – Summer 2014											
Grade	**6**	**5.5**	**5**	**4.5**	**4**	**3.5**	**3**	**2.5**	**2**	**1.5**	**1.0**
Number of Marks	75	67.5	60	52.5	45	37.5	30	22.5	15	7.5	0
Number of Students	2	2	1	4	1	5	3	3	1	1	0