Written Matura Examinations, Summer 2011
Kantonsschule Im Lee, Winterthur

4g N,MN,M immersive
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Mathematics Matura Examination

- You have 3 hours for 8 problems with 50 marks.
- You may use a graphing calculator without CAS and a formula booklet.
- Please show all your working.
- If possible, give exact values for your numerical answers.
  Otherwise, round your results appropriately.
- Solve each problem on a separate sheet of paper.

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1 [4m] The two normals to the curve of $y = \cos(2x)$ at the points $A\left(\frac{\pi}{4}, 0\right)$ and $B\left(\frac{3\pi}{4}, 0\right)$ intersect at point *C*. Find the coordinates of point *C*.

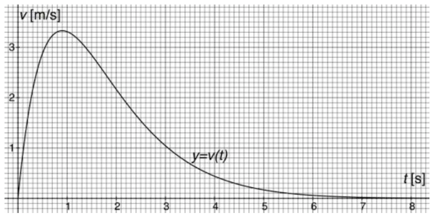
2 [7m] $y = a \cdot (x+b) \cdot (x-b)$ is the equation of a parabola which passes through the point $(1,1)$ and for which the area enclosed between the parabola and the *x*-axis is minimal. Find an equation of the parabola if $a < 0$.

3 [5m] Find the sum to infinity of the geometric series $S = \sum_{n=101}^{\infty} e^{-0.05n}$.

Show that $\int_{101}^{\infty} \left(e^{-0.05x}\right) dx$ is a good approximation for S by roughly sketching a graph and by evaluating the integral with calculus methods.

4 [4m] The graph shows the velocity v [in ms^{-1}] of a particle for $0 \le t \le 8$ seconds.

a) Read from the graph the velocity of the particle at $t = 0$, $t = 1$, $t = 2$ etc. and hence, using an appropriate approximation method of your own choice, find the total distance of the particle covered for $0 \le t \le 8$.



b) Find the average velocity of the particle for $0 \le t \le 8$.

c) Use the graph to find a rough approximation of the value of the greatest decrease in speed of the particle for $0 \le t \le 8$.

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5 [6m] Let x_0 be the zero (or *x*-intersect) of $f(x) = \frac{\ln(2x)}{5x^3} + 1$. Apply *one* step of Newton's method with initial approximation $x_1 = 1.0$ to find x_2. Then, with the help of your graphing calculator, draw the graph of *f* and give a (better) value of x_0. In your graph, show the positions of x_0, x_1 and x_2, explaining your result for x_2.

6 [13m] The points $A(-4,-10,0)$ and $B(2,2,4)$ are vertices of a triangle *ABC*.

a) Find a simplified equation of the plane which contains all vertices *C* such that the angle at *A* is 90°.

b) Find a simplified equation of the sphere which contains all vertices *C* such that the angle at *C* is 90°.

c) Vertex *C* now moves along a line *l* through $(0,6,0)$ in such a way that the area of the triangle *ABC* remains constant.

- **i)** Find an equation of line *l*.
- **ii)** Find the distance between the origin $(0,0,0)$ and the plane, which contains the triangle(s) *ABC*.

d) The equation $7\sqrt{(x-2)^2 + (y-2)^2 + (z-4)^2} + 6x + 12y + 4z - 52 = 0$ describes the locus of all vertices $C(x,y,z)$ such that the angle at vertex *B* is 60°.
(…der geometrische Ort aller Ecken *C*, so dass der Dreieckswinkel β 60° beträgt.)

- **i)** Show that the above statement is true.
- **ii)** Give a geometrical interpretation of the equation.
- **iii)** Find the coordinates of all such vertices *C*, which lie on the *x*-axis.

7 [5m] In the last 36 years 8 major accidents with nuclear power reactors have been recorded worldwide. Assume that the probability of an accident happening in any year is constant over the years.

a) Find the probability that there will be at least 4 major accidents in the next 10 years.

b) As of 2011, there are 637 nuclear power reactors worldwide. Find the probability that in the next 10 years no major accident will happen in any of the 5 Swiss reactors, assuming that the probability of such an accident is equally high for each one of the 637 reactors and that the number of reactors worldwide remains the same.

8 [6m] At the Wimbledon Men's Tennis Tournament, matches are best-of-five sets, i.e. sets are played until the first player wins 3 sets. Let X be the number of sets played in a match between two equally strong players.

a) Find the probability that the match ends in 3 sets.

b) Show that $P(X=4) = P(X=5)$.

Use the results of the problems in parts **a)** and **b)** to find

c) the expected number of sets in the match,

d) the standard deviation of *X*.

End of Examination

Solutions – Matura Examination in Mathematics, Summer 2011 – 4g N,MN,M immersive

1 $y' = -2\cdot\sin(2x)$ [1↓]; $y'(\frac{\pi}{4}) = -2 \Rightarrow m_{\text{normal}} = \frac{1}{2}$ [2↓]; normal line $y = \frac{1}{2}\left(x - \frac{\pi}{4}\right)$ [3↓];

symmetry: $x_c = \frac{\pi}{2} \Rightarrow$ point $\boxed{C\left(\frac{\pi}{2},\frac{\pi}{8}\right)}$ [4P]

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2 Point $(1,1)$: $1 = a\cdot(1+b)\cdot(1-b) \Rightarrow a = \frac{1}{1-b^2}$ [1P]; Area $A = 2\cdot\int_0^b a\left(x^2-b^2\right)dx$ [1↓];

$\Rightarrow A = 2a\cdot\left[\frac{1}{3}x^3 - b^2x\right]_0^b$ [2↓] $= -\frac{4}{3}ab^3$ [3↓]; $A(b) = \frac{4b^3}{3(b^2-1)}$ minimum [4↓]

$A'(b) = \frac{4}{3}\cdot\frac{3b^2\cdot(b^2-1) - 2b\cdot b^3}{\left(b^2-1\right)^2} = 0$ [5↓] $\Rightarrow b^2\left(b^2-3\right) = 0 \Rightarrow b = \pm\sqrt{3}$ [6↓] (or $b = 0$)

$\Rightarrow a = -\frac{1}{2} \Rightarrow y = -0.5\cdot\left(x+\sqrt{3}\right)\cdot\left(x-\sqrt{3}\right)$ or $\boxed{y = -0.5x^2 + 1.5}$ [7P]

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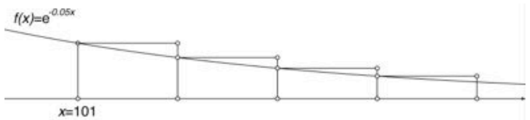
3 Ratio $r = e^{-0.05} \approx 0.9512$ [1↓]; sum to infinity $= \frac{a_1}{1-r} = \frac{e^{-0.05\cdot 101}}{1-e^{-0.05}} \approx \boxed{0.1314}$ [2P];

$\int_{101}^{\infty}\left(e^{-0.05x}\right)dx = -\frac{1}{0.05}\left(e^{-0.05x}\right)\Big|_{101}^{\infty} = -\frac{1}{0.05}\left(0 - e^{-0.05\cdot 101}\right) \approx \boxed{0.1282}$ [2P]



graph [1P]; Total: [5P]

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4 a)

t	0	1	2	3	4	5	6	7	8
v	0	3.3	2.2	1.05	0.45	0.2	0.1	0	0

; $s = \int_0^8 v(t)\,dt$ Simpson Approximation [1↓]

$s = \frac{8-0}{3\cdot 8}(0 + 4\cdot 3.3 + 2\cdot 2.2 + 4\cdot 1.05 + 2\cdot 0.45 + 4\cdot 0.2 + 2\cdot 0.1 + 4\cdot 0 + 0) \approx \boxed{7.9\text{ m}}$ [2P]

(or trapeziums: $s = \frac{8-0}{2\cdot 8}(0 + 2\cdot 3.3 + 2\cdot 2.2 + 2\cdot 1 + 2\cdot 0.4 + 2\cdot 0.2 + 2\cdot 0.1 + 0) \approx 7.2$ m)

b) $v_{\text{average}} = \frac{s}{t} \approx \frac{7.9}{8} \approx \boxed{1\text{ ms}^{-1}}$ [1P] **c)** slope at point of inflection $\approx \boxed{-1.36\text{ ms}^{-2}}$ [1P]

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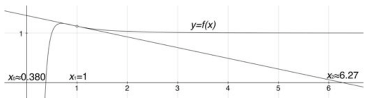
5 $f'(x) = \frac{1}{5}\cdot\frac{\frac{1}{x}\cdot x^3 - 3x^2\cdot\ln(2x)}{x^6} = \frac{1-3\cdot\ln(2x)}{5x^4}$ [2↓]; $x_2 = 1 - \frac{\frac{\ln(2)}{5}+1}{\frac{1-3\cdot\ln(2)}{5}}$ [3↓] $\approx \boxed{6.27}$ [4P]



; graph of f and $x_0 \approx 0.380$: [1P]; x_1, x_2: [1P]

(or $f_2(x) = \ln(2x) + 5x^3 = 0$; $x_2 = 1 - \frac{\ln 2 + 5}{1+15} \approx 0.644$)

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6 a) Plane normal to $\overrightarrow{AB} = \begin{pmatrix}6\\12\\4\end{pmatrix}$ [1↓]; $\vec{n} = \begin{pmatrix}3\\6\\2\end{pmatrix} \Rightarrow \boxed{3x+6y+2z+72=0}$ [2P]

b) $M(-1,-4,2)$; $r = \frac{1}{2}\left|\overrightarrow{AB}\right| = 7$ [1↓] $\Rightarrow \boxed{(x+1)^2+(y+4)^2+(z-2)^2 = 49}$ [2P]

c) Line parallel to AB through $(0,6,0)$: $\boxed{\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}0\\6\\0\end{pmatrix} + t\cdot\begin{pmatrix}6\\12\\4\end{pmatrix}}$ [1P];

$\overrightarrow{BA}\times\overrightarrow{BC} = \begin{pmatrix}-6\\-12\\-4\end{pmatrix}\times\begin{pmatrix}-2\\4\\-4\end{pmatrix} = \begin{pmatrix}64\\-16\\-48\end{pmatrix}$ [1↓]; plane ABC: $4x - y - 3z + 6 = 0$ [2↓];

distance $h = \frac{0+0+0+6}{\sqrt{4^2+(-1)^2+(-3)^2}} = \boxed{\frac{3}{13}\cdot\sqrt{26} \approx 1.177}$ [3P];

d) $\frac{\overrightarrow{BA}\circ\overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\cdot\left|\overrightarrow{BC}\right|} = \cos(\beta) \Rightarrow \frac{\begin{pmatrix}-6\\-12\\-4\end{pmatrix}\circ\begin{pmatrix}x-2\\y-2\\z-4\end{pmatrix}}{14\cdot\sqrt{(x-2)^2+(y-4)^2+(z-4)^2}} = \cos(60°)$ [1↓]

$\Rightarrow -6x - 12y - 4y + 52 = \frac{1}{2}\cdot 14\cdot\sqrt{(x-2)^2+(y-4)^2+(z-4)^2}$ [2P]

The equation describes an infinite $\boxed{\text{cone}}$ with vertex at B. [1P]

$C(x,0,0)$: $7\sqrt{(x-2)^2+4+16} + 6x - 52 = 0 \Rightarrow 13x^2 + 428x - 1528 = 0$

$\Rightarrow x_1 = 3.2494$; $x_2 = -36.1725 \Rightarrow \boxed{C_1(3.25,0,0);\ C_2(-36.17,0,0)}$ [2P]

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7 a) Binomial Distribution: $n = 10$, $p = \frac{8}{36} = \frac{2}{9}$ [1↓]: $1 - P(X \le 3)$ [2↓] $= \boxed{0.1632}$ [3P]

b) $\left(1 - \frac{2}{9}\cdot\frac{5}{637}\right)^{10} \approx \boxed{0.98}$ [2P] (or $\left(1 - \frac{2}{9\cdot 637}\right)^{50} \approx 0.98$ or $\left(\sqrt[637]{1-\frac{2}{9}}\right)^{50} \approx 0.98$)

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8 a) $P(X=3) = 2\cdot\left(\frac{1}{2}\right)^3 = \boxed{0.25}$ [1P] **b)** X=5: Each player wins 2 sets out of the first 4:

$P(X=5) = \binom{4}{2}\cdot\left(\frac{1}{2}\right)^2\cdot\left(\frac{1}{2}\right)^2$ [1↓] $= \frac{3}{8}$; $P(X=4) = 1 - \frac{2}{8} - \frac{3}{8} = \frac{3}{8} = P(X=5)$ [2P]

c) $E(X) = 3\cdot\frac{2}{8} + 4\cdot\frac{3}{8} + 5\cdot\frac{3}{8} = \boxed{\frac{33}{8} = 4.125}$ [1P] **d)** $V(X) = E\left(X^2\right) - \left(E(X)\right)^2 =$

$= 3^2\cdot\frac{2}{8} + 4^2\cdot\frac{3}{8} + 5^2\cdot\frac{3}{8} - \left(\frac{33}{8}\right)^2 = \frac{39}{64} \approx 0.609 \Rightarrow \sigma = \sqrt{V(X)} = \boxed{\frac{\sqrt{39}}{8} \approx 0.78}$ [2P]

Grading Scale and Results – 4g N,MN,M – Summer 2011

Grade	6	5.5	5	4.5	4	3.5	3	2.5	2	1.5	1.0
Number of marks	40	36	32	28	24	20	16	12	8	4	0
Number of students	2	2	1	1	3	4	3	0	0	0	0